MORE ADO ABOUT NOTHING ①

Theorem 1. Any local event that can happen in any arbitrary state of a field can also happen in the vacuum.

Theorem 2 In the vacuum any measurement located at x is maximally correlated with some simultaneous measurement located at y, however far apart x and y may be.

Theorem 3 Every local measurement is infinitely ambiguous, i.e. leaves infinitely many questions unanswered

NON-RELATIVISTIC QUANTUM FIELD THEORY (NRQFT)

Quantization of the Schrödinger field: At $t=0$,

$$\psi(\underset{\sim}{x}) = \frac{1}{(2\pi)^{3/2}} \int a(\underset{\sim}{k}) e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} d^3\underset{\sim}{x}$$

$$\psi^*(\underset{\sim}{x}) = \frac{1}{(2\pi)^{3/2}} \int a^*(\underset{\sim}{k}) e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}} d^3\underset{\sim}{x}$$

$$[a(\underset{\sim}{k}), a^*(\underset{\sim}{k}')] = \delta(\underset{\sim}{k}-\underset{\sim}{k}')$$

$$[\psi(\underset{\sim}{x}), \psi^*(\underset{\sim}{x}')] = \delta(\underset{\sim}{x}-\underset{\sim}{x}')$$

Define $N(\underset{\sim}{k}) = a^*(\underset{\sim}{k})\, a(\underset{\sim}{k})$

$$N(\underset{\sim}{x}) = \psi^*(\underset{\sim}{x})\, \psi(x)$$

$$N = \int N(\underset{\sim}{k})\, d^3\underset{\sim}{k} = \int N(\underset{\sim}{x})\, d^3\underset{\sim}{x}$$

$$N_V = \int_V N(\underset{\sim}{x})\, d^3\underset{\sim}{x}.$$

Then: $[N_V, N_{V'}] = 0$, for disjoint V and V'

N_V has eigenvalues $0, 1, 2, \ldots$

$N=0 \Rightarrow N_V = 0$ for all subvolumes V

so global vacuum $\Rightarrow$ local vacuum

RELATIVISTIC QUANTUM FIELD THEORY (RQFT)

Consider charged Klein-Gordon field

At $t=0$,

$$\phi(\underset{\sim}{x}) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\underset{\sim}{k}}{\omega(\underset{\sim}{k})} \left(a(\underset{\sim}{k}) e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} + b^*(\underset{\sim}{k}) e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}} \right)$$

$$\omega(\underset{\sim}{k}) = \sqrt{m^2 + k^2}$$

$$\left[a(\underset{\sim}{k}), a^*(\underset{\sim}{k}') \right] = \omega(\underset{\sim}{k})\, \delta(\underset{\sim}{k} - \underset{\sim}{k}')$$

$$N^+ = \int \frac{d^3\underset{\sim}{k}}{\omega(\underset{\sim}{k})} a^*(\underset{\sim}{k})\, a(\underset{\sim}{k}) \quad \text{.. Particles}$$

$$N^- = \int \frac{d^3\underset{\sim}{k}}{\omega(\underset{\sim}{k})} b^*(\underset{\sim}{k})\, b(\underset{\sim}{k}) \quad \text{... Antiparticles}$$

But if we write $N_V^+ = \int_V N(\underset{\sim}{x})\, d^3\underset{\sim}{x}$

we find $\left[N_V^+, N_{V'}^+ \right] \neq 0$ for disjoint V and V'.

Similarly for N_V^-.

So $N_V^\pm$ does not commute with $N = N^+ + N^-$

and the global vacuum $N = 0$

$$\not\Rightarrow N_V^\pm = 0.$$

Two ways out

(1) N_V^+ is not a local observable

we need quantities like Q_V, the charge in volume V, for which $[Q_V, Q_{V'}] = 0$ for disjoint V and V'

But again $N = 0 \not\Rightarrow Q_V = 0$

(since $[Q_V, N] \neq 0$)

$\longrightarrow$ local charge fluctuations.

(2) Write $N_V^+ = \int \underline{N_{NW}^+}(\xi)\, d\xi$ Newton-Wigner

where $N_{NW}^+(\underset{\sim}{\xi}) = \psi_{NW}^*(\underset{\sim}{\xi})\, \psi_{NW}(\underset{\sim}{\xi})$

and $\psi_{NW}(\underset{\sim}{\xi}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\underset{\sim}{k}}{\omega(\underset{\sim}{k})}\, a(\underset{\sim}{k})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{\xi}} \cdot \sqrt{\omega(\underset{\sim}{k})}$

Then

$$[\psi_{NW}(\underset{\sim}{\xi}), \psi_{NW}^*(\underset{\sim}{\xi}')] = \delta(\underset{\sim}{\xi} - \underset{\sim}{\xi}')$$

and hence

$$\left[N^{+}_{V,\,NW} \,,\; N^{+}_{V',\,NW} \right] = 0$$

for disjoint V and V'.

<u>But</u> $\psi^{*}_{NW}(\underset{\sim}{\xi})$ cannot be interpreted as creating a particle localized at the point ξ, because we also have

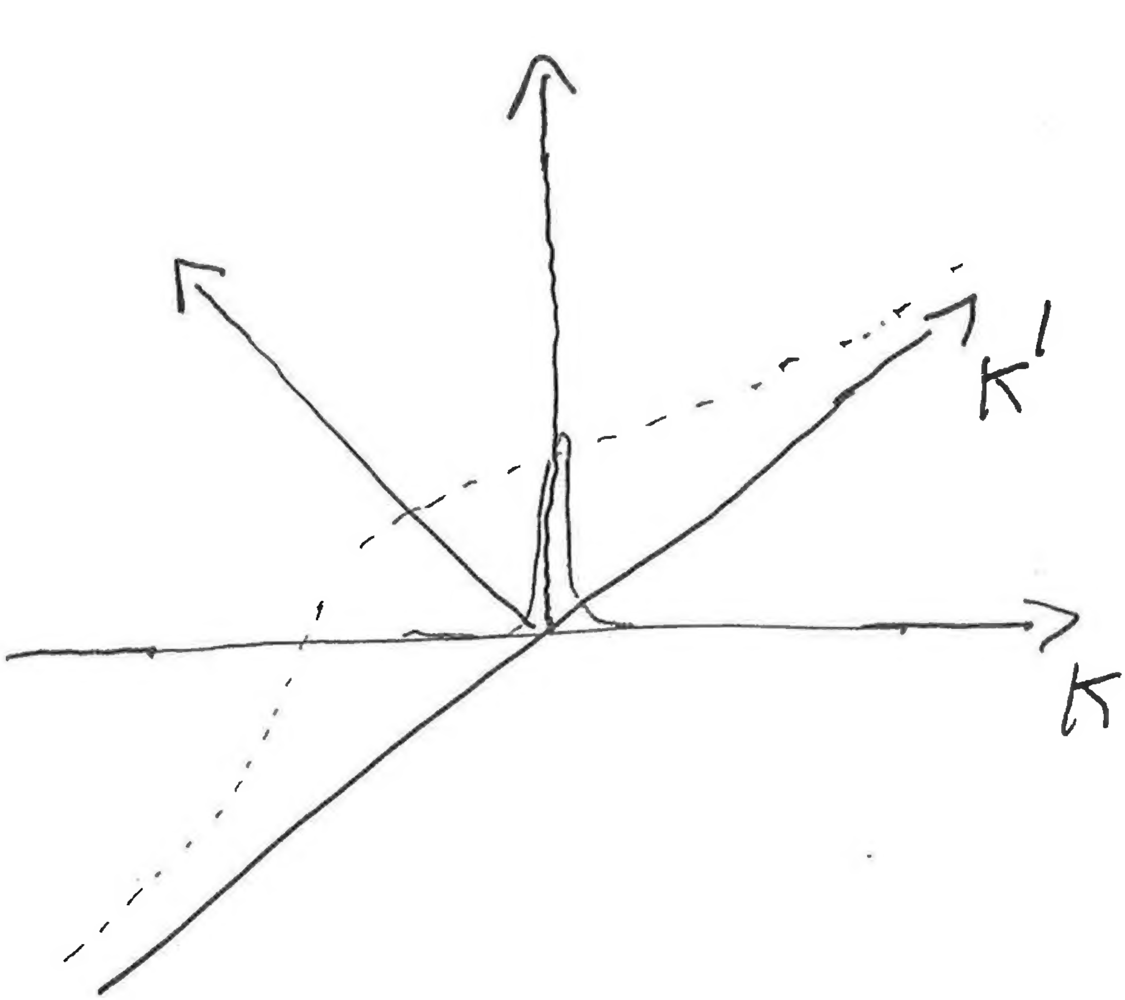
$$\langle \psi^{*}_{NW}(\underset{\sim}{\xi})\Omega \,|\, Q(\underset{\sim}{x}) \,|\, \psi^{*}_{NW}(\underset{\sim}{\xi})\Omega \rangle \neq 0$$

for $\underset{\sim}{\xi} \neq \underset{\sim}{x}$.

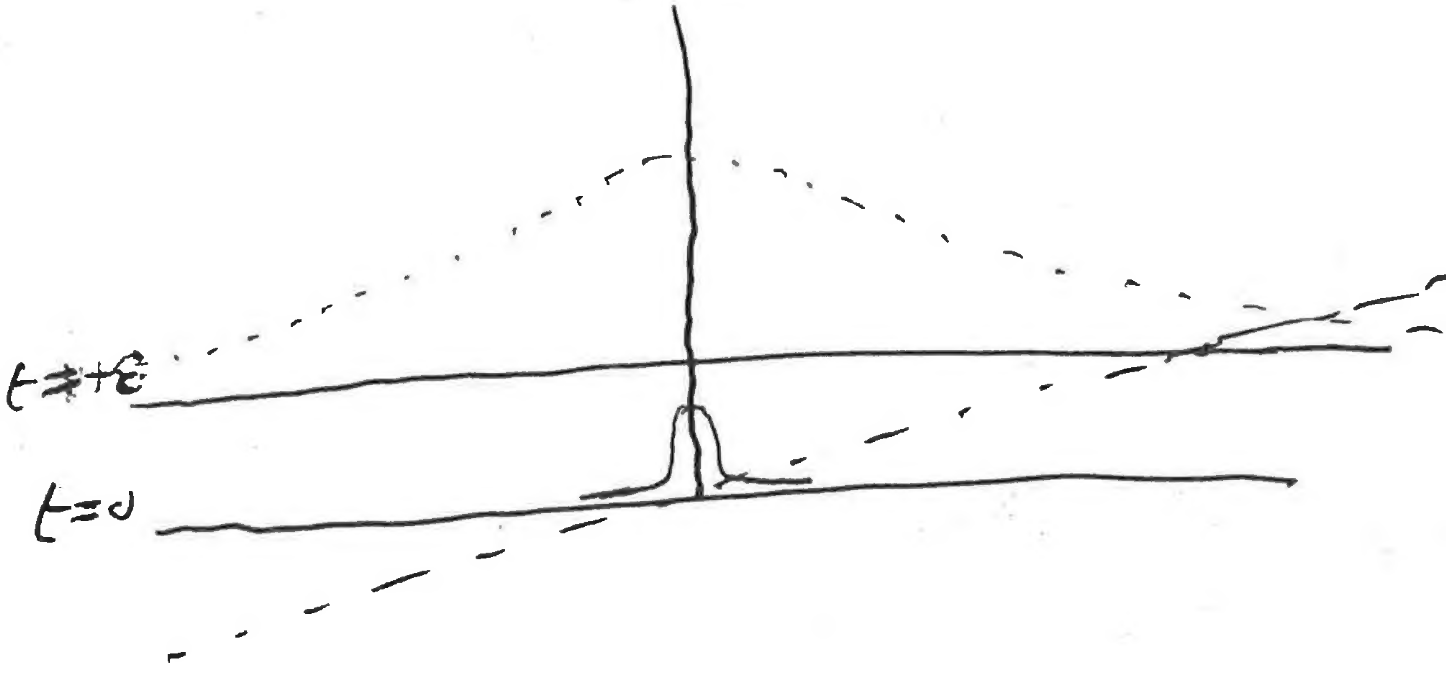
(Ω is the vacuum state)

NW localization is 'spread out' in physical $\underset{\sim}{x}$-space.

This arises because NW states are nonlocalized on inclined hyperplanes



This diagram explains
Hegerfeldt's Paradox (1974)



ALGEBRAIC QUANTUM FIELD THEORY

$$O \longmapsto R(O)$$

└ bounded open set in space time

Von Neumann algebra of observables

R acts on Hilbert space $\mathcal{H}$

$$R(O + \underline{a}) = U(\underline{a})\, R(O)\, U^*(\underline{a})$$

└ representation of translation $\underline{x} \to \underline{x} + \underline{a}$

For time like translations $U(\underline{a})$ is exponentiated to obtain a Hamiltonian operator which is non-negative.

<u>Isotony</u> For any two bounded open sets O, O', $O_1 \subseteq O_2 \Rightarrow R(O_1) \subseteq R(O_2)$

<u>Locality</u> If O_1 and O_2 are spacelike related, then $\forall A_1 \in R(O_1)$, $\forall A_2 \in R(O_2)$

$$[A_1, A_2] = 0$$

The Global algebra R is the smallest von N. algebra containing all the local algebras. We assume that the representation of R is irreducible and generated by the translates of $R(O)$ for any O.

<u>The Vacuum</u> Ω is the unique state which is invariant under all translations

<u>The Reeh- Schlieder Theorem</u>

Ω is <u>cyclic</u> with respect to H for <u>any</u> $R(O)$

This just means $\{A\Omega : A \in R(O)\}$ is dense in H.

Corollary

Ω is a separating vector for $R(0)$

This just means

$$\forall A \in R(0), \quad A\Omega = 0 \Rightarrow A = 0.$$

I am now going to prove the R-S theorem and its corollary for a very simple analogue of a field theory, in which spacetime collapses to two points and the Von N. algebras are just the algebras of operators on a 2-dimensional Hilbert space. This is just the familiar 2 spin ½ particle system, and for the analogue of the vacuum we shall take initially

$$\Psi_{\text{singlet}} = \frac{1}{\sqrt{2}}\Big(|\sigma_{1z}=+1\rangle \otimes |\sigma_{2z}=-1\rangle$$
$$- |\sigma_{1z}=-1\rangle \otimes |\sigma_{2z}=+1\rangle \Big)$$

Then $\forall \phi \in \mathcal{H}_1 \otimes \mathcal{H}_2$, $\exists A_1 \in R_1$ s.t.

$$|\phi\rangle = A_1 |\Psi_{\text{singlet}}\rangle$$

Proof: By inspection.

$$\text{If } |\phi\rangle = \alpha\, |\sigma_{1z}=+1\rangle \otimes |\sigma_{2z}=-1\rangle$$
$$+ \beta\, |\sigma_{1z}=-1\rangle \otimes |\sigma_{2z}=-1\rangle$$
$$+ \gamma\, |\sigma_{1z}=-1\rangle \otimes |\sigma_{2z}=+1\rangle$$
$$+ \delta\, |\sigma_{1z}=-1\rangle \otimes |\sigma_{2z}=+1\rangle$$

Then $A_1 = \alpha P_1^+ + \beta Q_1 P_1^+ + \gamma P_1^-$
$+ \delta\, \gamma\, Q_1 P_1^-$

where $P_1^{\pm}$ projects the state $|\sigma_{1z}=\pm 1\rangle$
and Q_1 rotates spin 1 thro' $180°$.

Similarly $\forall \phi \in \mathcal{H}_1 \otimes \mathcal{H}_2$, $\exists A_2 \in R_2$ s.t.
$|\phi\rangle = A_2 |\Psi_{\text{singlet}}\rangle$

Corollary

$$A_1 \,|\Psi_{\text{singlet}}\rangle = 0 \implies A_1 = 0$$

Proof:

By the baby R-S theorem $\forall \phi \in \mathcal{H}_1 \otimes \mathcal{H}_2$, we can write

$$|\phi\rangle = A_2 \,|\Psi_{\text{singlet}}\rangle ,$$ so

$$\begin{aligned} A_1 |\phi\rangle &= A_1 A_2 \,|\Psi_{\text{singlet}}\rangle \\ &= A_2 A_1 \,|\Psi_{\text{singlet}}\rangle \\ &= 0 \end{aligned}$$

Since ϕ is any vector in $\mathcal{H}_1 \otimes \mathcal{H}_2$, it follows that $A_1 = 0$, Q.E.D.

So $|\Psi_{\text{singlet}}\rangle$ is a cyclic vector and a separating vector for R_1 (and similarly for R_2).

We now prove a baby version of Theorem 1

Define $p = \text{Prob}^{\Psi_{\text{singlet}}}\left(P_1 \in R_1 = 1\right)$
projector.

then $p = \| P_1 |\Psi_{\text{singlet}}\rangle \|^2$

So $p = 0 \Rightarrow P_1 |\Psi_{\text{singlet}}\rangle = 0$

$\Rightarrow P_1 = 0$ (by R-S)

$\therefore P_1 \neq 0 \Rightarrow p \neq 0.$ QED.

We now turn to a baby version of Theorem 2

We want to prove.

$\forall P_2, \exists P_1$ s.t. $\langle P_1 P_2 \rangle_{\Psi_{\text{singlet}}}$

$= \langle P_1 \rangle_{\Psi_{\text{singlet}}}$

(i.e. $\text{Prob}^{\Psi_{\text{singlet}}}(P_2 = 1 / P_1 = 1) = 1$)

<u>Proof</u> Write $\Psi_{\text{singlet}} = \psi_s$

Write $|\phi\rangle = P_2|\psi_s\rangle / \| P_2|\psi_s\rangle \|$

Then by construction

$$\langle P_2 \rangle_\phi = 1 \quad \text{———} \quad (1)$$

But, by the baby R-S theorem

$$|\phi\rangle = c_1 |\psi_s\rangle \quad \text{———} \quad (2)$$

where c_1 is some operator on $\mathcal{H}_1$ (extended to $\mathcal{H}_1 \otimes \mathcal{H}_2$)

Substituting (2) in (1) gives

$$\langle \psi_s | Q_1 P_2 | \psi_s \rangle = 1 \quad --- \quad (3)$$

where $Q_1 = c_1^* c_1$ is a positive Hermitian operator on $\mathcal{H}_1$.

So we can expand

$$Q_1 = \lambda_1 \Gamma_1 + \lambda_1' \Gamma_1' \quad --- \quad (4)$$

where λ_1, λ_1' are the positive real eigenvalues of Q_1, and Γ_1, Γ_1' are orthogonal projections in $\mathcal{H}_1$.

Substituting (4) in (3) yields (14)

$$w_1 \frac{\langle \Gamma_1 P_2 \rangle_{\psi_s}}{\langle \Gamma_1 \rangle_{\psi_s}} + w_2 \frac{\langle \Gamma_1' P_2 \rangle_{\psi_s}}{\langle \Gamma_1' \rangle_{\psi_s}}$$

$$= 1 \quad --- (5)$$

where

$$w_1 = \lambda_1 \langle \Gamma_1 \rangle_{\psi_s}$$

$$w_2 = \lambda_1' \langle \Gamma_1' \rangle_{\psi_s}$$

But we know

$$\langle Q \rangle_{\psi_s} = \| C_1 |\psi_s\rangle \|^2$$

$$= \| |\phi\rangle \|^2 = 1$$

$$\therefore \quad w_1 + w_2 = 1 \quad --- (6)$$

with $w_1 \geq 0$, $w_2 \geq 0$.

Hence LHS (5) $\leq \max\left(\frac{\langle \Gamma_1 P_2 \rangle_{\psi_s}}{\langle \Gamma_1 \rangle_{\psi_s}}, \frac{\langle \Gamma_1' P_2 \rangle_{\psi_s}}{\langle \Gamma_1' \rangle_{\psi_s}} \right)$

and (5) can only be satisfied if Γ_1 or Γ_1' (or both) satisfy the condition for Theorem 2 Q.E.D.

Now Theorems 1 and 2 are trivially true for $\Psi_{singlet}$.

Theorem 1 just says, all spin components have non-vanishing probability for results ± 1 on either particle (indeed for $\Psi_{singlet}$ all the probabilities are equal to 1/2!)

while Theorem 2 says all spin components ~~are~~ on one particle are maximally correlated with spin components on the other particle. (indeed these are just the mirror-image correlations of $\Psi_{singlet}$!)

But the proofs of these well-known results for $\Psi_{singlet}$ only used the R–S theorem; so they can be lifted straight back to QFT, with the vacuum replacing $\Psi_{singlet}$!

In the QFT case, Theorem 2 can be formulated more accurately as:

For any two spacelike separated bounded open regions O_1 and O_2 and $\forall \varepsilon > 0$, $\forall P_2 \in R(O_2)$

$\exists P_1 \in R(O_1)$ s.t.

$$\langle P_1 P_2 \rangle_\Omega \geq (1-\varepsilon) \langle P_1 \rangle_\Omega$$

We can also express the maximality of the correlations specified in Theorem 2 in terms of correlation coefficients. For any two projectors P_1 and P_2 belonging to $R(O_1)$ and $R(O_2)$ respectively, we have

$$C(P_1, P_2) = \frac{\langle P_1 P_2 \rangle - \langle P_1 \rangle \cdot \langle P_2 \rangle}{\left[\langle P_1 \rangle \cdot (1 - \langle P_1 \rangle) \cdot \langle P_2 \rangle \cdot (1 - \langle P_2 \rangle)\right]^{\frac{1}{2}}}$$

So, for fixed $\langle P_1 \rangle$, $\langle P_2 \rangle$ the maximum value of $C(P_1, P_2)$ is given by

$$C^{max}(P_1, P_2) = \left[\frac{\langle P_1 \rangle \cdot (1 - \langle P_2 \rangle)}{\langle P_2 \rangle \cdot (1 - \langle P_1 \rangle)} \right]^{1/2} \quad \text{--(1)}$$

This only attains the value 1 when $\langle P_1 \rangle = \langle P_2 \rangle$

This condition is satisfied for $|\Psi_{singlet}\rangle$, but Theorem 2 in no way depends on this condition.

We now want to compare (1) with the well-known Fredenhagen bound on correlation coefficients (Fredenhagen 1985).

This reads

$$C(P_1, P_2) \leq e^{-ml} \sqrt{(1 - \langle P_1 \rangle_\Omega) \cdot (1 - \langle P_2 \rangle_\Omega)} \quad \text{--(2)}$$

where m is mass-gap, and l the minimum

Comparing (1) and (2), consistency requires

$$\langle P_1 \rangle_\Omega \leq \frac{e^{-2m\ell} \langle P_2 \rangle_\Omega}{\left(1 - \langle P_2 \rangle_\Omega\right)^2} \quad \cdots (3)$$

i.e. ~~the~~ for a fixed value of $\langle P_2 \rangle_\Omega$, the maximally correlated P_1 must have a probability of occurring that falls off exponentially with the distance between O_1 and O_2.

This result shows how difficult it would be to observe the long-range correlations in the vacuum. But, of course, it does not show that they don't exist!

Turning to Theorem 3, the ambiguity referred to arises in the $\Psi_{singlet}$ case from the fact that the local projectors are all two-dimensional (i.e. of the form $P_1 \otimes I_2$ etc)

In QFT the technical formulation of Theorem 3 is:

$\forall P \in R(O)$, P is infinite-dimensional

<u>Proof</u> By Driessler's Theorem (1975) the Von N. algebra associated with an unbounded wedge in spacetime is a type III factor.

But every bounded open region is contained in some wedge.

So, by isotony, $R(O)$ is always a sub-algebra of a type III factor.

But in a type III factor all the projectors are infinite-dimensional.

Hence all the projectors in $R(O)$ are infinite-dimensional

Q.E.D.

N.B. This result does not demonstrate that every local algebra is type III — this still remains an open question.

As a corollary of Theorem 3 we can state:

It is never a local question to ask

" Are we in the vacuum state or indeed in an N-particle state (i.e. orthogonal to the vacuum)? "

This raises the fundamental question:

What do (local) particle detectors detect?

The answer is they cannot strictly speaking be detecting particles. They detect certain types of field excitation, which for all practical purposes → FAPP may resemble particles.

But in reality (if you will excuse the phrase!) QFT is not a theory of particles, but a theory of fields and their local excitations, and that is all there is to it.
